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IS THERE A HELMHOLTZ MIXING COEFFICIENT?

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PHYSICS

ABSTRACT

In the slip stream behind a Mach shock one has two layers of gas of the same material set into sliding motion with respect to each other under conditions where all initial turbulence can be eliminated. Photographs of such shocks reveal that an intermingling of the two layers takes place, which in the first approximation grows linearly with the amount of the displacement of the one layer with respect to the other. The present report attempts to determine a "wiping coefficient" which represents the ratio of the depth of the layer of mixing to the lateral displacement of the two layers with respect to each other. Contrary to expectation, no universal value for this "wiping coefficient" was found in the four cases for which data were available:

Gas	Shock Strength	Wiping Coefficient
CCl_4	3.73	0.20
CCl_4	8.21	0.36
CO_2	1.68	~ 0
Air	4.06	0.11

For this reason it is concluded that many more such events must be analyzed before one can form a satisfactory picture of the mixing processes. The methods of calculation applicable for this purpose are also contained in the text.

I. INTRODUCTION.

When two fluids slide over each other, their interface becomes the seat of an instability analyzed by Helmholtz.[1] Any periodic very small departure of the interface from planarity, of the form

$$\delta y = a(t) \sin kx,$$

will grow exponentially,

$$\frac{d^2 a}{dt^2} = \alpha^2 a(t),$$

with a growth constant, α , which is proportional to the wave number:

$$\alpha = k \frac{(\rho_1 \rho_2)^{1/2}}{\rho_1 + \rho_2} \Delta U,$$

where ΔU is the difference in velocity of the two fluids. The reason for this instability is illustrated in Figure 1. After the disturbance reaches an amplitude equal to some small fraction of the wave length the interface commences to change its form and the process takes on a quiet non-linear character. Some attempt has been made by L. Rosenhead (Proc. Roy. Soc. A, 170, 1931) to study mathematically the initial phases of this non-linear growth process with the results here duplicated from his Figure 4.

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[1]See Horace Lamb, Hydrodynamics, 6th ed., pp. 22 and 373, Cambridge University Press, 1932 (Dover reprint, New York, 1945).

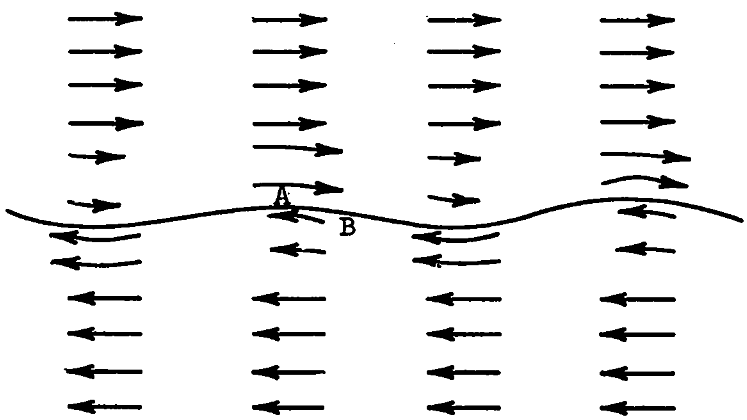
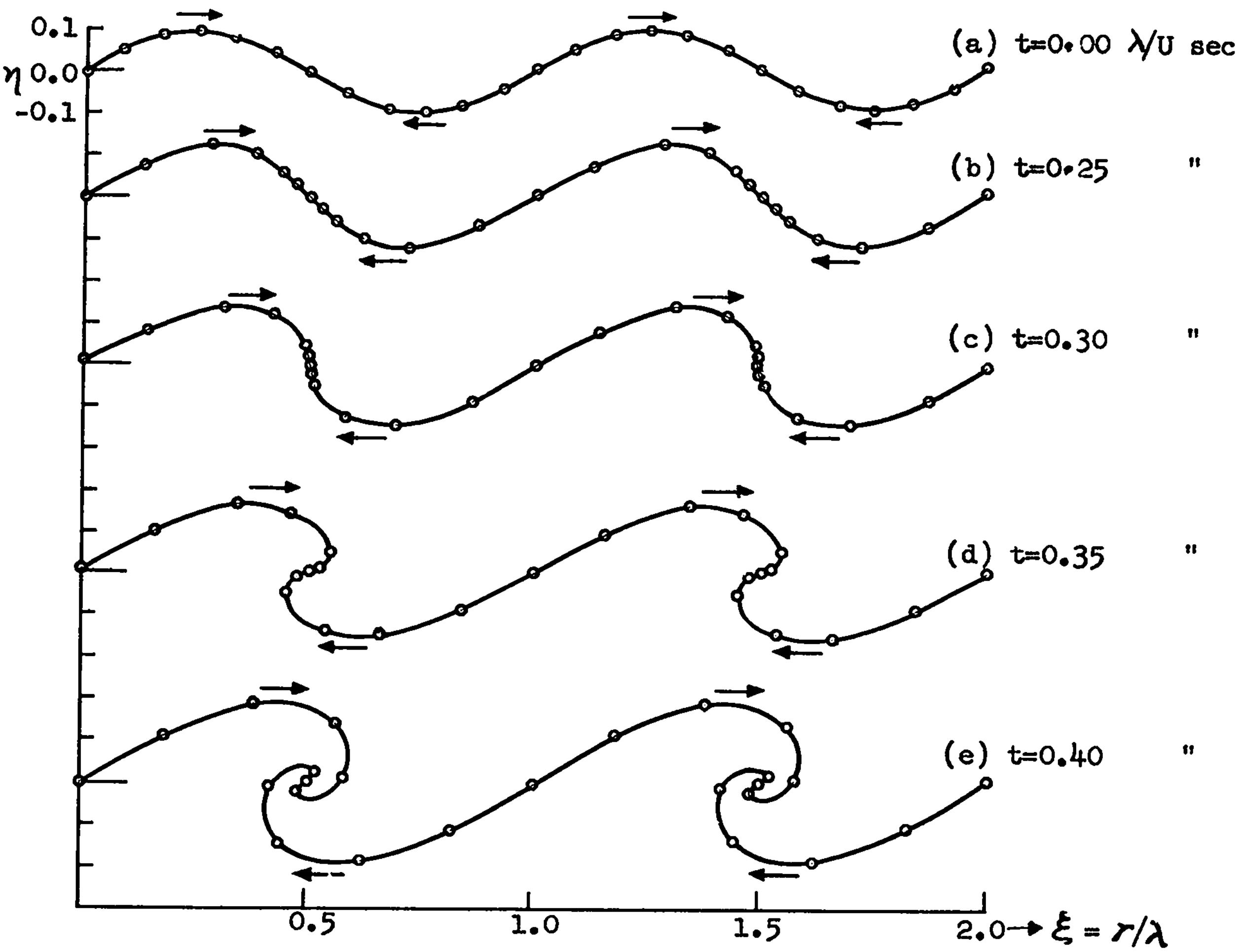


Figure 1. Mechanism of growth of a small irregularity in the interface between two fluids in lateral motion with respect to each other. At point A the cross section available for the motion of the upper fluid is less than normal. Consequently, the fluid moves with a greater than normal motion as indicated by the slightly longer arrow. By Bernoulli's principle, the pressure in this region is, therefore, less than normal. For this reason the lower fluid is sucked up still further in the direction of region A. Consequently, the disturbance grows with time.



Rosenhead's Figure 4.

The further growth of the instability is so complicated that it appears desirable--in default of theoretical understanding of the further stages of the mixing--to secure what empirical evidence one can on the process.[2] For this purpose the slip stream of a Mach reflection, Figure 2, offers an ideal means to set two layers of gas into motion relative to each other without initial turbulence. The layers of gas on either side of the slip stream have been raised to the same pressure, one by a single shock, the other by two shocks. The compression in the latter case is the more nearly adiabatic. Consequently, the entropy change for the singly shocked gas is greater. The temperature higher and the pressures being equal, the density is lower. The difference in density and difference in relative velocity of the two layers of gas on either side of the slip stream is proportional for weak shocks to the cube of the shock strength, but can become quite sizable for strong shocks.

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[2]See, however, the thesis, "Turbulent Mixing in Two-Dimensional Flows," by Albert I. Bellin, Department of Engineering Sciences and Applied Physics, Harvard University, September 1947; also, Tollmien, "Berechnung turbulenter Ausbreitungsvorgange," ZAMM, 4, p. 468, 1926.

Note added later: We are kindly informed by Dr. E. Frieman, who has analyzed the theoretical curves of Rosenhead, which are shown in his Figure 4, that the ratio

$$\frac{\text{depth of "mixing"}}{\text{distance of lateral motion}} = \text{wiping coefficient,}$$

that is found from these curves, is approximately 0.2, to be compared with the four empirical values of the wiping coefficient which are listed in the Abstract. For a fuller discussion of the physics of this mixing, see a report of E. Frieman now in course of preparation.

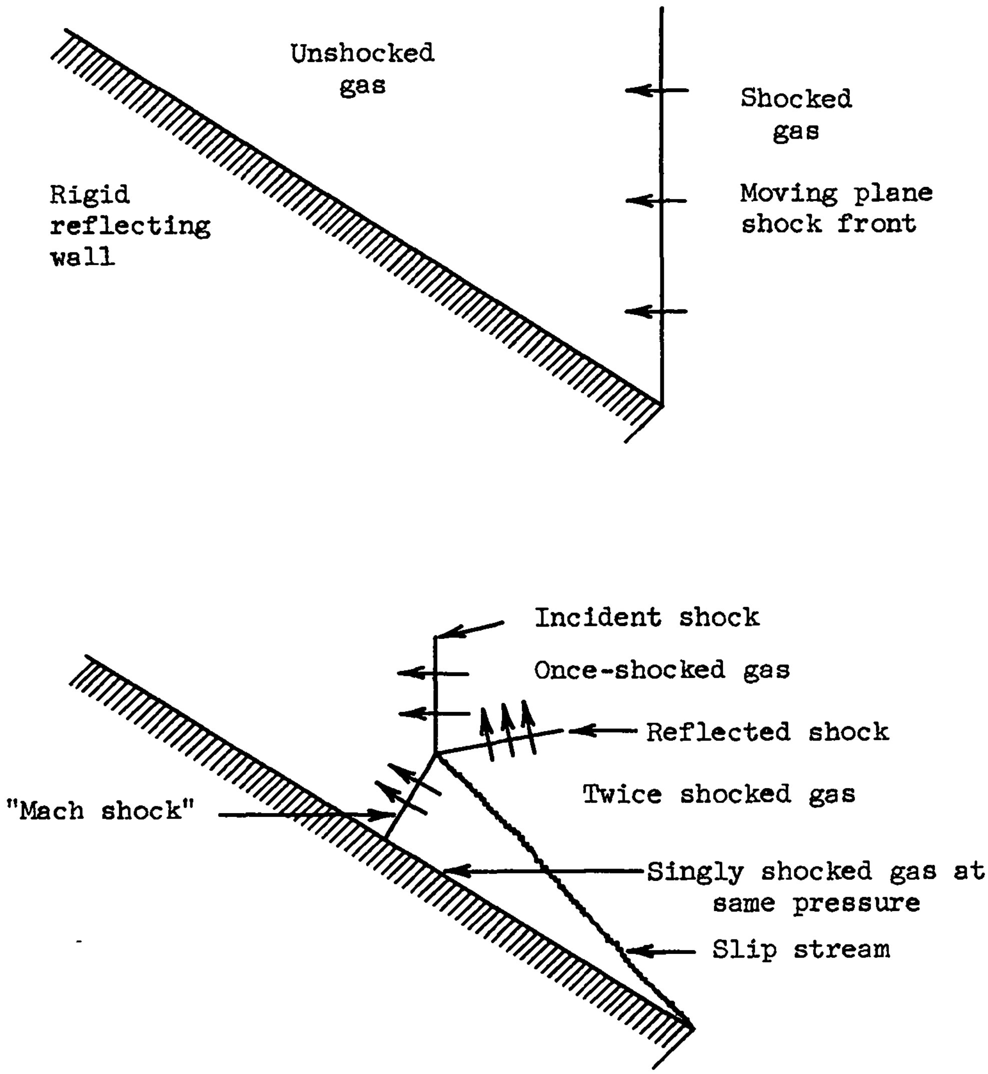

Figure 2. Early (above) and later (below) stages in shock phenomena initiated by plane shock incident obliquely on a rigid reflecting wall. Wavy line drawn for the slip stream is meant to symbolize the turbulence shown at this layer on most actual photographs.

Photographs show that the demarkation between the two gas layers is not ordinarily sharply defined but manifests a wavy irregular appearance such as corresponds to the presence of turbulence. Moreover, the width of this zone of mixing is generally seen to grow approximately linearly, at least at first, with distance from the triple point of the shock. Later there are other perturbations and the linear growth law may fail to hold or the slip stream may get quite curled up. These complications are presumably due at least in part to wall effects and are left out of account in the present analysis.[3]

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[3]We are greatly indebted to Dr. R. E. Duff of the Los Alamos Scientific Laboratory for supplying us with the three photographs, made while he was at the University of Michigan, of two slip streams in carbon tetrachloride and one slip stream in carbon dioxide; and to Professor Walter Bleakney of Princeton University for supplying us with a beautiful photograph of the slip stream in air, used in working out the data for the fourth case listed in the Abstract.

II. THE WIPING FUNCTION.

As a quantitative measure of the amplitude of the Helmholtz instability at a point relative to the slide of the fluid on one side of the slip stream over the fluid on the other side at that point, we define the function $I(x)$, where x is the distance on S measured from T, the triple point.

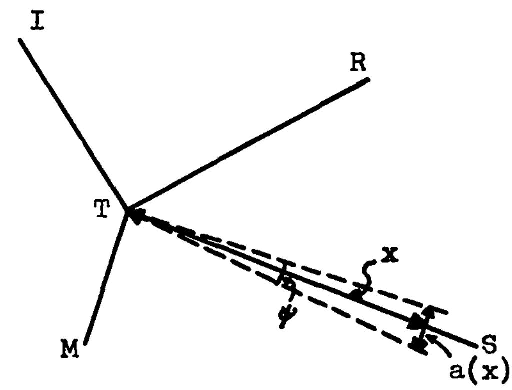

Figure 3.

$$I(x) \equiv \frac{a(x)}{\Delta(x)} \tag{2.1}$$

where $a(x)$ is the width of the instability at x (see Figure 3), and $\Delta(x)$ is the difference of the distances traveled by fluid particles along S in the upper fluid (in the region RTS) and lower fluid (region MTS) respectively in the time taken for the lower fluid particle to travel from T to the point of coordinate x.

$a(x)$ can be measured from photographs. $\Delta(x)$, however, must be calculated from the initial data (angle and strength of the incident shock) via some theory of plane shock reflections. For this purpose we use the simple theory [1] in which the various physical variables are assumed to vary discontinuously across the shocks and be constant in the various angular regions shown in Figure 3. In this theory the problem in analysis is replaced by the algebraic one of satisfying the Rankine-

Hugoniot equations across the various shocks.

In the notation of [1] , where unprimed, primed, double primed and subscript 1 denote the physical variables in the angular regions ITM, ITR, RTS, and MTS, respectively, we get the following equations for the determination of $\Delta(x)$:

$$d_1 \equiv x = z_1 t_1, \qquad d'' = z'' t_1, \qquad \Delta(x) = d'' - d_1$$

where, since the flow in some neighborhood of T is parallel to S, we are concerned with the magnitudes z_1, z'' of the flow velocities in regions MTS and RTS respectively, and t_1 is the time taken by the lower fluid particle to go from T to x. Eliminating t_1,

$$(2.2) \qquad \Delta(x) = (R-1)x, \quad R \equiv z''/z_1$$

hence

$$(2.3) \qquad I(x) = \frac{1}{R-1} \cdot \frac{a(x)}{x}$$

From qualitative considerations [2] $R > 1$. If $\frac{a(x)}{x}$ is small and nearly constant, as is usually the case, $\frac{a(x)}{x} \cong \psi$, where ψ is the "instability" angle in radian measure (see Figure 3). To this approximation, I becomes the number

$$(2.3)' \qquad I = \frac{1}{R-1} \cdot \psi \quad .$$

III. CALCULATION OF R.

In Figure 4 is represented the flow of the fluid (streamlines indicated by dotted lines) as it comes in at the left in the direction $\vec{Z}$, is "refracted" by the shock configuration, and goes off to the right in the direction $\overrightarrow{TS}$. The shock configuration (lines of physical discontinuity) is indicated by heavy solid lines. The angles between the normals to the shocks and the various lines of flow are indicated by τ, $\tau+\delta$, etc.

The flux of fluid incident upon the shock configuration between the line $\vec{Z}$ and the upper dotted line, at distance $s = 1$ from $\vec{Z}$, must be equal by mass conservation to the flux of fluid away from the shock configuration between the line $\overrightarrow{TS}$ and the upper dotted line, at distance say s'' from $\overrightarrow{TS}$. A similar equation holds between the incident flux below $\vec{Z}$ and the outgoing flux below $\overrightarrow{TS}$. Hence the outgoing fluxes between $\overrightarrow{TS}$ and the upper and lower dotted lines, at distances from it s'' and s_1, respectively, are equal. Expressing these equations:

$$\rho'' s'' z'' = \rho_1 s_1 z_1$$

hence

$$R \equiv z''/z_1 = \frac{\rho_1}{\rho''} \cdot \frac{s_1}{s''} \tag{3.1}$$

Now where $\eta \equiv \rho'/\rho$, $\eta' \equiv \rho''/\rho'$, $\eta_1 \equiv \rho_1/\rho$ in the notation of [1],

$$\rho_1/\rho'' = \frac{\eta_1}{\eta\eta'} \tag{3.2}$$

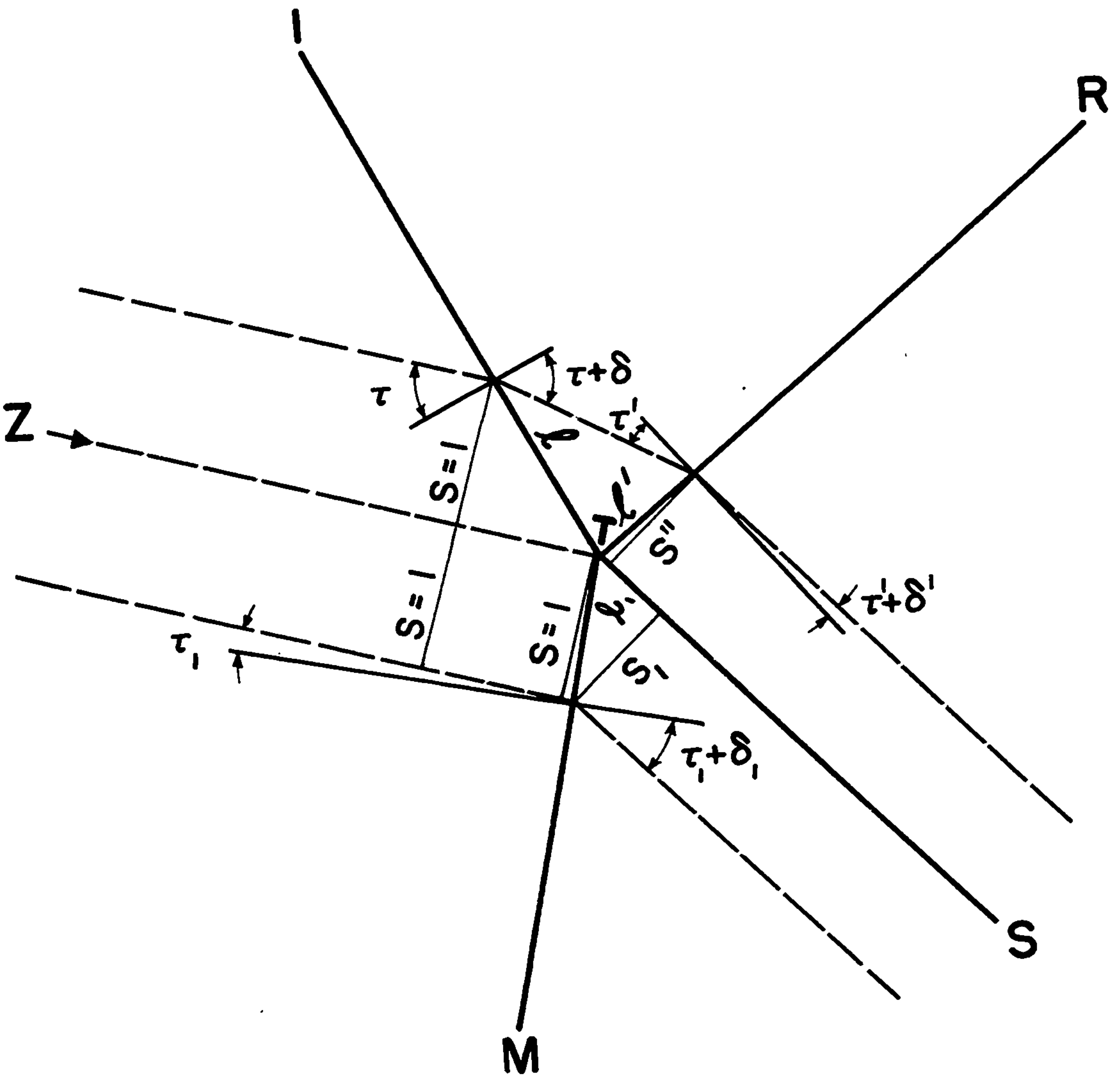

Figure 4. Shock configuration showing the streamlines and angles relevant to the calculation of R.

As to the other factor, we have

$$\ell \cos \tau = s = 1 \qquad \frac{\ell}{\cos \tau'} = \frac{\ell'}{\cos(\tau+\delta)}$$
$$\ell' \cos(\tau'+\delta') = s''$$

the last equation by the law of sines in the triangle of sides ℓ, ℓ' and the segment of the streamline between TI and TR (cf. Figure 4). Similarly

$$\ell_1 \cos \tau_1 = s = 1, \qquad \ell_1 \cos(\tau_1+\delta_1) = s_1 .$$

Eliminating ℓ, ℓ' and ℓ_1 from these equations, we get

$$\text{(3.3)} \qquad \frac{s_1}{s''} = \frac{\cos \tau \cos \tau' \cos(\tau_1+\delta_1)}{\cos(\tau+\delta)\cos(\tau'+\delta')\cos \tau_1} \equiv$$

Therefore

$$\text{(3.4)} \qquad R = \frac{\eta_1}{\eta \eta'} \cdot \frac{\cos \tau \cos \tau' \cos(\tau_1+\delta_1)}{\cos(\tau+\delta)\cos(\tau'+\delta')\cos \tau_1}$$

Given the angle T and some measure of the incident shock strength, say $y \equiv p'/p$, R can be computed from the theory [1] . Some of the laborious calculations can be avoided, however, by determining some of the relevant quantities directly by measurements of angles from the photographs. Let us assume that for a given fluid of specific heat ratio γ we are given $\xi (\equiv p/p')$ and also that we can determine $-\vec{Z}$, the vector of triple point motion, from the photograph. Then we can determine from angle measurements on the photographs

ω $\equiv$ angle between the incident shock and $-\vec{Z}$

ω' $\equiv$ angle between $\vec{Z}$ and the reflected shock.

ζ $\equiv$ angle between $-\vec{Z}$ and the normal to the Mach shock.

ϵ $\equiv$ angle between $\overrightarrow{TS}$ (slip stream) and $\vec{Z}$.

All these angles are positive in the counterclockwise sense. Here the notation of L. Smith [3] is used; the relation between these angles and those shown in Figure 4 is easily determined by inspection or by referring to his Figure 14. [He also gives tables exhibiting the empirical correlation for air between α and α', the angles made by I and R, respectively, with the reflecting plate, and ω and ω' for shocks of various strengths. This permits the determination of ω and ω' in the case that the motion of the triple point is not determinable from the photograph.] Then a suggested computational program for determining R is

(3.5) a. $$\eta = \frac{(\gamma+1)y + (\gamma-1)}{(\gamma+1) + (\gamma-1)y}$$

b. $$x = \cot \omega$$

c. $$\tan \theta \equiv \tan(\omega+\omega')$$

d. $$x' = -\frac{\tan\theta + \eta x}{1-\eta x \tan\theta}$$

e. $$B^2 = \frac{1+(\eta x)^2}{1+(x')^2}$$

f. $$\eta' = \frac{(\gamma+1)B^2}{(\gamma-1)(B^2-1) + (\gamma+1)\eta}$$

g. $\eta_1 = (\gamma+1) \dfrac{2(B^2-1) + \gamma+1 - (\gamma-1)\eta}{2(\gamma-1)(B^2-1) + (\gamma+1)\left\{\gamma+1 - (\gamma-1)\eta\right\}}$

h. $B = \sqrt{B^2}$

i. $\beta = B \cdot \dfrac{\cos(\zeta+\epsilon)\sin\omega}{\cos\zeta\,\sin(\omega'+\epsilon)}$

j. $R = \dfrac{\eta_1}{\eta'\eta} \cdot \beta$

The wiping coefficient I, eq. (2.3)', was computed in this way from four shock tube photographs showing Helmholtz instability of the slip stream. The results are tabulated below.

Photo. #	Gas, γ	y	τ°	R	ψ	I
1215	CCL_4, 1.13	3.73	32	1.25	.05	.198
1216	CCL_4, 1.13	8.21	39.5	1.28	.10	.360
1209	CO_2, 1.304	1.68	31.3	1.11	~ 0	~ 0
3411	Air, 1.400	4.06	44	1.953	.105	.110

The first three pictures were taken by Russell Duff at Michigan; the last, by Bleakney at Princeton. Note that for #1209, no instability was perceptible ($\psi \cong 0$), whence I = 0.

IV. THEORETICAL WIPING COEFFICIENT.

As a point of interest, we show in this section that the existence of a wiping coefficient (I independent of x) cannot be understood on the basis of the linear theory.[1] That is, it is presumably a feature of the asymptotic behavior.

The solution of the linearized theory corresponding to stream velocities U' and U in the upper and lower fluids respectively, and an initial interface

$$\eta\big|_{t=0} = a \cos k\xi$$

with initial velocity μ at the nodes:[4]

$$\dot{\eta}\big|_{t=0} = \mu \sin k\xi$$

is

$$\eta = a \cos k(Vt-\xi)\cosh\alpha t + \left(\frac{kVa-\mu}{\alpha}\right) \sin k(Vt-\xi)\sinh\alpha t$$

$$V \equiv \frac{\rho U+\rho' U'}{\rho+\rho'}\,, \qquad \alpha \equiv \frac{(\rho\rho')^{1/2}}{\rho+\rho'}\,\Delta Uk, \qquad \Delta U \equiv |U-U'| \tag{4.1}$$

The solution (4.1) describes a train of running waves moving in the positive ξ direction with constant velocity V whose amplitudes grow exponentially at a rate governed by the magnitude of α. In defining I at any fixed point $\xi = x$, consider the trajectory of the point (ξ,η) which is $(0,a)$ at $t = 0$, (see Figure 5). This latter point is

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[4] We include the case $\mu \neq 0$ for complete generality; the condition $\mu = 0$, or the interface is initially at rest, is usual.

the crest of an initial wave. In time $T = x/V$ this crest will have reached the point directly over $\xi = x$. Twice the excess of the amplitude of the wave at this point over the initial amplitude a divided by the "slide" $\Delta(x)$ of the upper fluid over the lower in the time $\bar{T} \equiv x/U$ taken by a lower fluid particle to go from $\xi = 0$ to $\xi = x$ gives us I(x) by definition. In symbols

$$I(x) = \frac{2\left[\eta(\xi = x, t = T) - \eta(\xi = 0, t = 0)\right]}{U \cdot \bar{T}} \tag{4.2}$$

Substituting for T and $\bar{T}$ and using the definition of R, equation (2.2), this gives

$$I(x) = \frac{2a(\cosh \alpha x/V - 1)}{(R-1)x} \equiv \frac{4a \sinh^2 \frac{\alpha x}{2V}}{(R-1)x} \tag{4.3}$$

Formula (4.3) shows immediately that no wiping coefficient exists, for the series expansion of I(x) has no constant term:

$$I(x) = \frac{a\alpha^2}{V^2(R-1)} \cdot x + \cdots \tag{4.4}$$

If we write $I(x) = 1/R-1\ \psi(x)$ as in equation (2.3') , then for small x, $\psi(x)$ is linear in x. Thus an instability angle proportional to x and thus vanishing at x = 0 would be predicted by the linear theory. The instability region would have flaring rather than straight line sides, terminating in a cusp at x = 0 (see Figure 6). Since the photographs seem to indicate straight line sides and a non-zero angle at the

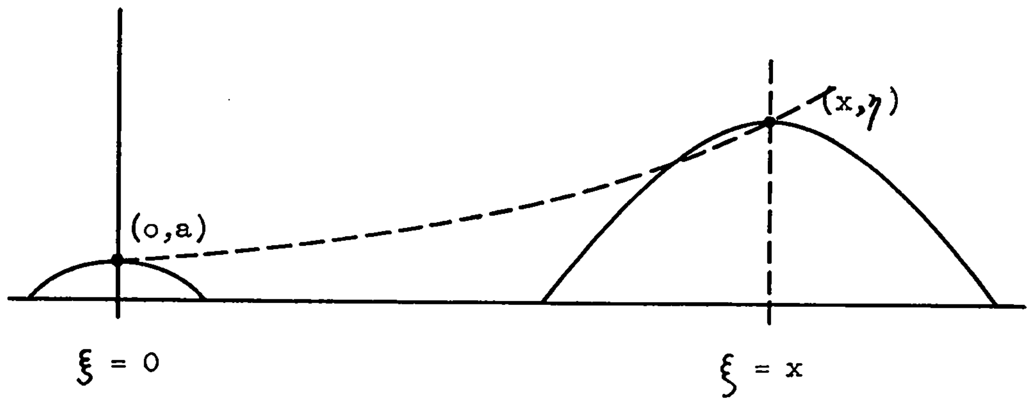

Figure 5.
Evolution of an interfacial running wave according to the linear theory of Helmholtz instability.

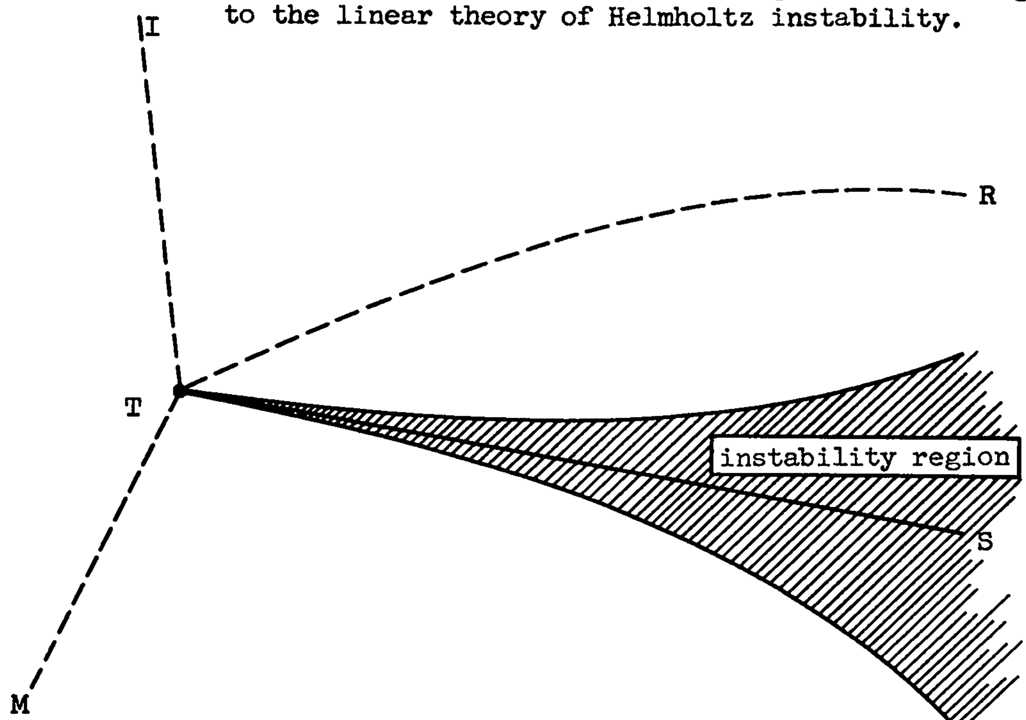

Figure 6.
The instability region as it would appear for $I(x) \propto x$.

triple point, we conclude that in the observed phenomenon, the motion has got beyond the linear phase of Helmholtz mixing.[5]

V. BIBLIOGRAPHY

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2 J. von Neumann, Private communication.

3 Lincoln Smith, "Photographic Investigation of the Reflection of Plane Shocks in Air," OSRD Report No. 6271 (1945).

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[5] See also footnote 2 and a forthcoming report by E. Frieman supporting this view.